

ON CARLESON'S INEQUALITY II

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ABSTRACT. We present a new and simple proof of improved Carleson's inequality.

1. Introduction

The classical Hardy's inequality reads:

$$(1.1) \quad \int_0^\infty \left(\frac{1}{x} \int_0^x f(y) dy \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx,$$

where f is a nonnegative measurable function on $(0, \infty)$ and $p > 1$. A weighted modification of (1.1) was proved also by Hardy [4] as

$$(1.2) \quad \int_0^\infty \left(\frac{1}{x} \int_0^x f(y) dy \right)^p x^\alpha dx \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^\infty f^p(x) x^\alpha dx,$$

where f is a nonnegative measurable function on $(0, \infty)$, $p > 1$, and $\alpha < p - 1$. The constant

$$\left(\frac{p}{p-1-\alpha} \right)^p$$

is the best possible.

The importance and the usefulness of these inequalities could never have been overestimated. As a limit versions of Hardy's inequality, the following inequality appeared in the literature. See [5] and [9].

THEOREM A (Polya-Knopp inequality).

$$(1.3) \quad \int_0^\infty \exp \left\{ \frac{1}{x} \int_0^x \ln f(t) dt \right\} dx \leq e \int_0^\infty f(x) dx$$

for any measurable $f \geq 0$.

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The discrete version of inequality (1.3) is the famous Carleman's inequality [2]. Being a limiting form of Hardy's inequality (1.1) and a continuous form of Carleman's inequality, (1.3) has attracted a lot of attention and its extensions have been investigated frequently. See for example [3] and the references therein. Among extensions of (1.3) is the following result of L. Carleson.

THEOREM B (Carleson's inequality). Let $F(x)$ be a convex function for $x \geq 0$, satisfying $F(0) = 0$. If $-1 < \alpha < \infty$, the following inequality holds true:

$$(1.4) \quad \int_0^{\infty} x^{\alpha} e^{-F(x)/x} dx \leq e^{\alpha+1} \int_0^{\infty} x^{\alpha} e^{-F'(x)} dx.$$

See [1]. Theorem B gives, for any measurable $f \geq 0$, that

$$(1.5) \quad \int_0^{\infty} x^{\alpha} \exp \left\{ \frac{1}{x} \int_0^x \ln f(t) dt \right\} dx \leq e^{\alpha+1} \int_0^{\infty} f(x) x^{\alpha} dx,$$

whence Theorem B extends Theorem A. See [7]. So, (1.3) is a special case of (1.5), and (1.5) can be regarded as a special case of (1.4) as well as a limiting case of (1.2).

The goal of this note is to present a new and simple proof of the following improved version of Carleson's inequality.

THEOREM 1.1. Let $0 < b \leq \infty$. Let $F(x)$ be a real valued function, differentiable almost everywhere on $(0, b)$, continuous on $[0, b)$ with $F(0) = 0$. If $-1 < \alpha < \infty$, then the following inequality holds true:

$$(1.6) \quad \int_0^b x^{\alpha} e^{F(x)/x} dx \leq e^{\alpha+1} \int_0^b \left(1 - \frac{x}{b}\right) x^{\alpha} e^{F'(x)} dx.$$

As was shown in [1] and [9], the proofs of Carleson's inequality and its L^p version inequality there depended heavily on the convexity of F . The most striking fact of Theorem 1.1 may be that the convexity assumption on F in Theorem B is removed. Note that the convexity of F implies the continuity of F and the existence of F' almost everywhere. Instead, our proof depends on the following inequality, called the continuous form of Hölder's inequality (see [6, 8]).

THEOREM C (Continuous form of Hölder's inequality). Let $X = (X, \mu)$ be a probability measure space and $Y = (Y, \nu)$ be a positive σ -finite measure space. Let $f(x, y)$ be a positive measurable function on

$X \times Y$. Then

$$(1.7) \quad \int_Y \exp \left(\int_X \ln f(x, y) \, d\mu(x) \right) d\nu(y) \leq \exp \left\{ \int_X \ln \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x) \right\}.$$

Equality holds in (1.7) as a nonzero finite value if and only if $f(x, y) = g(x)h(y)$ almost everywhere $\mu \times \nu$ for a positive μ -measurable function g with $-\infty < \int_X \ln g \, d\mu < \infty$ and a positive ν -measurable h with $\int_Y h d\nu = 1$.

2. Proof of Theorem 1.1

Note that

$$(2.1) \quad \frac{F(x)}{x} = \int_0^1 F'(xy) \, dy, \quad x \in (0, b).$$

For simplicity, let $G = \exp F'$. Then by (2.1)

$$(2.2) \quad \int_0^b x^\alpha \exp \left\{ \frac{F(x)}{x} \right\} dx = \int_0^b x^\alpha \exp \left\{ \int_0^1 F'(xy) \, dy \right\} dx = \int_0^b x^\alpha \exp \left\{ \int_0^1 \ln G(xy) \, dy \right\} dx.$$

Applying (1.7) we obtain

$$(2.3) \quad \int_0^b x^\alpha \exp \left\{ \int_0^1 \ln G(xy) \, dy \right\} dx \leq \exp \left[\int_0^1 \ln \left(\int_0^b x^\alpha G(xy) dx \right) dy \right].$$

On the other hand, a straightforward calculation gives

$$(2.4) \quad \begin{aligned} & \exp \left[\int_0^1 \ln \left(\int_0^b x^\alpha G(xy) dx \right) dy \right] \\ &= \exp \left[\int_0^1 \ln \left(\frac{1}{y^{\alpha+1}} \int_0^{by} x^\alpha G(x) dx \right) dy \right] \\ &= e^{\alpha+1} \cdot \exp \left[\int_0^1 \ln \left(\int_0^{by} x^\alpha G(x) dx \right) dy \right]. \end{aligned}$$

By the convexity of the exponential function and by changing the order of the integration,

$$\begin{aligned}
 & \exp \left[\int_0^1 \ln \left(\int_0^{by} x^\alpha G(x) dx \right) dy \right] \\
 (2.5) \quad & \leq \int_0^1 \left(\int_0^{by} x^\alpha G(x) dx \right) dy \\
 & = \int_0^b x^\alpha G(x) \int_{x/b}^1 dy dx \\
 & = \int_0^b \left(1 - \frac{x}{b} \right) x^\alpha G(x) dx.
 \end{aligned}$$

Inequality (1.6) now follow from (2.1)~(2.5). The proof is complete.

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